Lecture 15

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1 Homogeneous systems

Last time we studied linear functions. To continue our theory about them we have to study another topic, which we will use in our future lectures.

Recall, that the system is called homogeneous, if it has 0's in the right hand side. The main fact about them is that each homogeneous system has either 1 or infinitely many solutions; moreover, if the number of equations is less than the number of variables, then the system has infinitely many solutions.

We'll state some properties of the solution set of the homogeneous system.

Existence of 0. $(0, 0, \dots, 0)$ is a solution.

Addition of solutions. Let we have 2 solutions of the homogeneous system, $(y_1, y_2, \dots, y_n)$ and $(y'_1, y'_2, \dots, y'_n)$. Then $(y_1 + y'_1, y_2 + y'_2, \dots, y_n + y'_n)$ is a solution. To check it consider an equation of the system:

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = 0.$$

Since specified n-tuples are solutions, we have:

$$a_{i1}y_1 + a_{i2}y_2 + \cdots + a_{in}y_n = 0 \quad \text{and}$$
$$a_{i1}y'_1 + a_{i2}y'_2 + \cdots + a_{in}y'_n = 0.$$

Adding these 2 equations, we have:

$$a_{i1}(y_1 + y'_1) + a_{i2}(y_2 + y'_2) + \cdots + a_{in}(y_n + y'_n) = 0,$$

so, $(y_1 + y'_1, y_2 + y'_2, \dots, y_n + y'_n)$ is a solution.

Multiplication by a number. Let $(y_1, y_2, \dots, y_n)$ be a solution. Then $(cy_1, cy_2, \dots, cy_n)$ is a solution. To check it consider an equation of the system:

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = 0.$$

Since specified n-tuples are solutions, we have:

$$a_{i1}y_1 + a_{i2}y_2 + \cdots + a_{in}y_n = 0$$

Multiplying equation by c, we get

$$a_{i1}(cy_1) + a_{i2}(cy_2) + \cdots + a_{in}(cy_n) = 0,$$

so $(cy_1, cy_2, \ldots, cy_n)$ is a solution.

So we see, that for any 2 solutions their sum is also a solution, zero is a solution, and for any solution, constant multiplied by this solution is a solution. So, we get the following theorem:

Theorem 1.1. *The solution set of the homogeneous system is a vector space.*

Let's analyze this vector space. I.e. we would like to find its dimension and basis. First we will show it in the example.

Example 1.2. *Consider the following system:*

$$\begin{cases} x_1 & + & 2x_2 & - & x_3 & + & 2x_4 & = & 0 \\ 2x_1 & + & 5x_2 & + & x_3 & - & x_4 & = & 0 \\ 3x_1 & + & 7x_2 & & & + & x_4 & = & 0 \end{cases}$$

Let's try to solve it. Subtracting the first equation from the second one and the third one we get:

$$\begin{cases} x_1 & + & 2x_2 & - & x_3 & + & 2x_4 & = & 0 \\ & & x_2 & + & 3x_3 & - & 3x_4 & = & 0 \\ & & x_2 & + & 3x_3 & - & 3x_4 & = & 0 \end{cases}$$

and the third equation is redundant. Free variables in this system are x_3 and x_4. Assigning values $x_3 = k_3$ and $x_4 = k_4$ we have:

$$x_2 = -3k_3 + 3k_4$$
$$x_1 = -2x_2 + x_3 - 2x_4 = 2(-3k_3 + 3k_4) + k_3 - 2k_4 = -5k_3 + 4k_4.$$

So, the solution set is:

$$\{(-5k_3 + 4k_4, -3k_3 + 3k_4, k_3, k_4) \mid k_3, k_4 \in \mathbb{R}\}$$

Let's get 2 particular solutions by assigning first $k_3 = 1, k_4 = 0$, and then $k_3 = 0, k_4 = 1$.

- $k_3 = 1, k_4 = 0$*: the solution is* $(-5, -3, 1, 0)$.
- $k_3 = 0, k_4 = 1$*: the solution is* $(4, 3, 0, 1)$.

Now we see, that any solution can be expressed as a linear combination of these 2 particular solutions:

$$(-5k_3 + 4k_4, -3k_3 + 3k_4, k_3, k_4) = k_3(-5, -3, 1, 0) + k_4(4, 3, 0, 1).$$

Moreover, these solutions are linearly independent. To check it let's form linear combination which is equal to $\mathbf{0}$*:*

$$x(-5, -3, 1, 0) + y(4, 3, 0, 1) = (0, 0, 0, 0).$$

Let's consider the third component and the fourth component. We'll have $x = 0$*, and* $y = 0$*. So, these vectors are linearly independent. Then these solutions form a basis in the solution space of the system, and dimension of the solution space is 2.*

Now we can state general result for homogeneous systems.

Theorem 1.3. *Let's suppose that the number of free variables in REF of the homogeneous system is equal to* s*. Let* $u_1, u_2, \dots, u_s$ *be the solutions obtained by setting one of the free variables equal to 1, and remaining free variables equal to 0. Then the dimension of the solution space is* s *and vectors* u_i*'s form a basis.*

Proof. To prove this theorem we have to first prove that these solutions span all solution space, and that they are independent.

Since the values of leading variables are defined by the free variables, we see, that solution which has first free variable equal to k_1, second free variable equal to k_2, etc. till s-th free variable equal to k_s is

$$k_1u_1 + k_2u_2 + \cdots + k_su_s$$

(since in u_1 the first free variable equals to 1, and all remaining free variables are equal to 0, in u_2 the second free variable equals to 1, and all remaining free variables are equal to 0, etc.)

Moreover, if $k_1u_1 + k_2u_2 + \cdots + k_3u_3 = 0$, then we have $k_1 = 0, k_2 = 0, \dots, k_s = 0$ (considering equation corresponding to the first free variable, the second free variable, etc.), and thus vectors are linearly independent.

So, they form a basis of the solution space, and its dimension is equal to the number of free variables. □

2 Image and kernel

Let we have a linear function: $f : U \to V$, where U and V are vector spaces. We will give 2 definitions associated with this function.

Definition 2.1. ***Image*** *of* f *is the set of vectors* v *from* V *for which there exists a vector* u *in* U *such that* $f(u) = v$*. I.e. it consists of images of all vectors from* U*. It is denoted by* $\operatorname{Im} f$*.*

Example 2.2. *Let's consider the projection function* $f : \mathbb{R}^3 \to \mathbb{R}^3$ *such that* $f(x, y, z) = (x, y, 0)$. *Then the set of all triplets with the third component equal to 0 is an image of* f*:*

$$\operatorname{Im} f = \{(x, y, 0) \mid x, y \in \mathbb{R}\}.$$

Definition 2.3. *The* ***kernel*** *of* f *is the set of all elements* u *from* U *such that* $f(u) = \mathbf{0}$. *It is denoted by* $\operatorname{Ker} f$.

Example 2.4. *Let's consider the projection function* $f : \mathbb{R}^3 \to \mathbb{R}^3$ *such that* $f(x, y, z) = (x, y, 0)$. *Then all vectors with first 2 coordinates which are equal to 0 and any third coordinate form a kernel of* f.

$$\operatorname{Ker} f = \{(0, 0, z) \mid z \in \mathbb{R}\},$$

since $f(0, 0, z) = (0, 0, 0)$ *for any* $z \in \mathbb{R}$.

We'll consider another example.

Example 2.5. *Let's consider a function of taking a derivative of a polynomial in the space* $\mathbb{P}_2$, *i.e.*

$$D(a_2x^2 + a_1x + a_0) = 2a_2x + a_1.$$

We see that image of this function consists of polynomials of degree 1 or 0, and the kernel consists of constants — polynomials of degree 0 (only their derivative is equal to 0):

$$\operatorname{Im} D = \mathbb{P}_1$$
$$\operatorname{Ker} D = \mathbb{R} = \mathbb{P}_0$$